Nikolai Kolachevsky

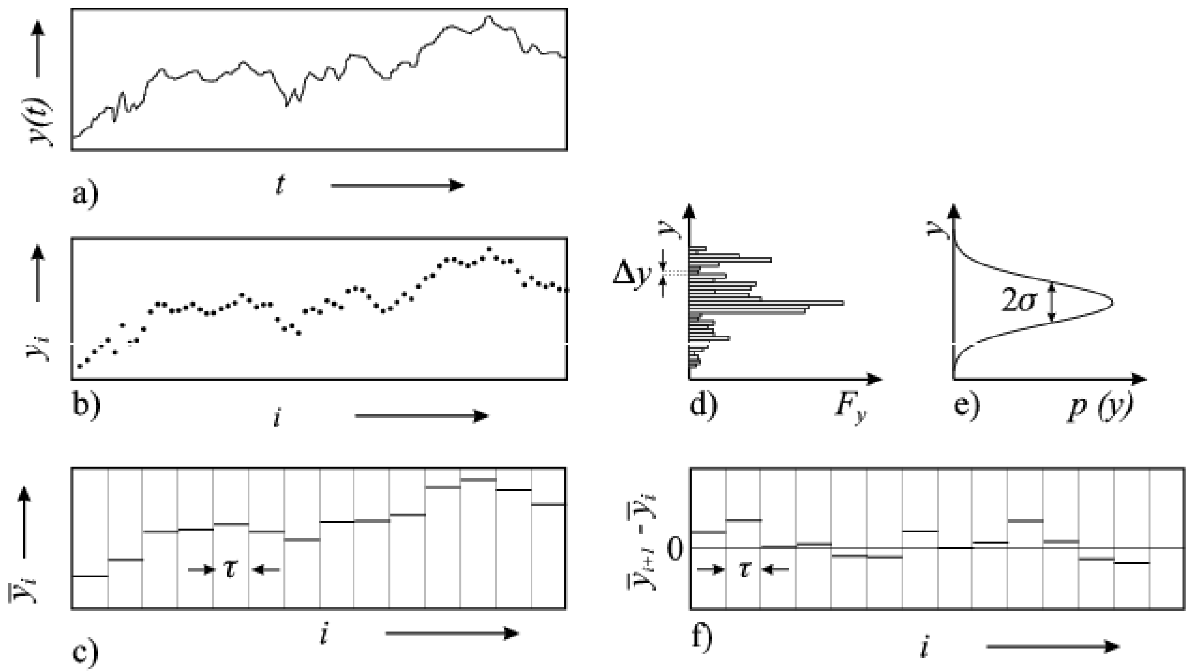
Lecture 2

- Mathematical description of stochastic processes.
- Allan deviation
- From spectral representation of fluctuations to time representation.
- Spectral density and Allan deviation of different fluctuation types.

Oscillator with amplitude/phase fluctuations

$$U(t) = [U_0 + \Delta U(t)] \cos(2\pi\nu_0 t + \phi(t))$$

Normalized phase and frequency $\quad x(t) \equiv \dfrac{\phi(t)}{2\pi\nu_0} \qquad y(t) \equiv \dfrac{\Delta\nu(t)}{\nu_0} = \dfrac{dx(t)}{dt}$



Mean value

$$\bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$$

Dispersion

$$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{y})^2$$

How to measure mean and dispersion?

In experiment one can only evaluate the mean value and dispersion

Stationary process: mean value and dispersion do not depend on time

By definition, the quantitative characteristics of any stochastic phenomenon is derived from an ensemble of atoms, e.g. in atomic gas, etc.

Very regularly it is impossible: one does not have a big number of systems (sometimes only one)

Ergodic process: time realization of one unit gives the same result as is an ensemble of units is studied

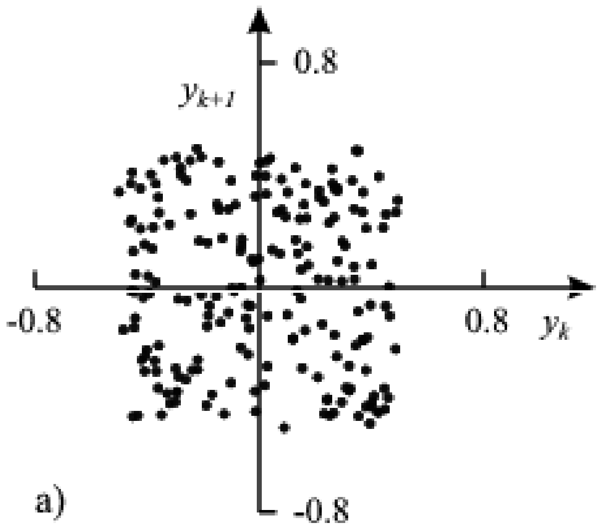
1		1(t)
2		1(t+t)
3	⇔	1(t+2t)
…		…
N		1(t+Nt)

Correlations?

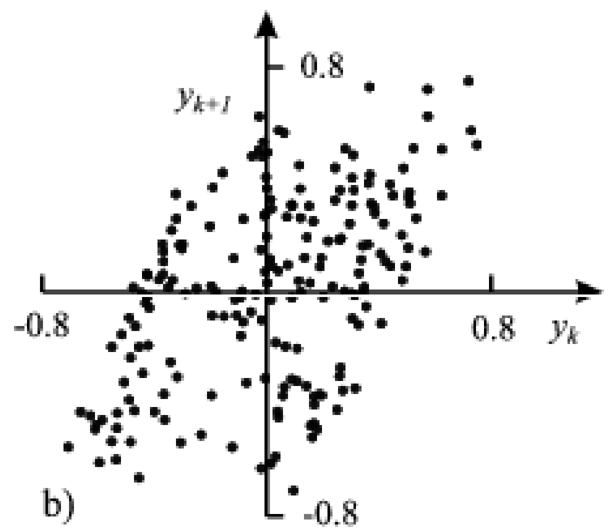
For correlated data one has to take in into account covariance terms

Correlated data $$y_{k+1} = \alpha y_k + \epsilon$$

Uncorrelated



Correlated



Allan deviation

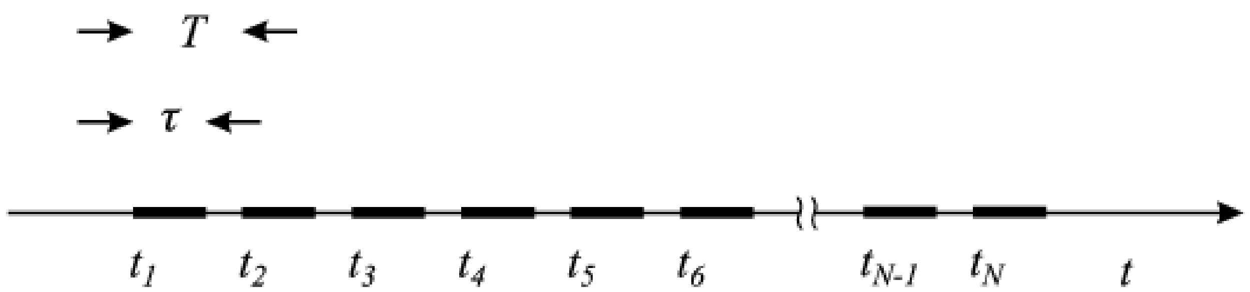
For non-stationary process its parameters can depend on time. E.g. if linear drift is present, the classical definition of dispersion gives divergent result => Allan dispersion!

$$\sigma_y^2(\tau) = \left\langle \sum_{i=1}^{2} \left(\bar{y}_i - \frac{1}{2} \sum_{j=1}^{2} \bar{y}_j \right)^2 \right\rangle = \frac{1}{2} \langle (\bar{y}_2 - \bar{y}_1)^2 \rangle$$

Difference of neighboring measurement readings!

$$\sigma_y^2(\tau) = \frac{1}{2\tau^2} \left\langle (\bar{x}_{i+2} - 2\bar{x}_{i+1} + \bar{x}_i)^2 \right\rangle$$

$$\bar{y}_i = \frac{\bar{x}_{i+1} - \bar{x}_i}{\tau}.$$



Example: linear drift

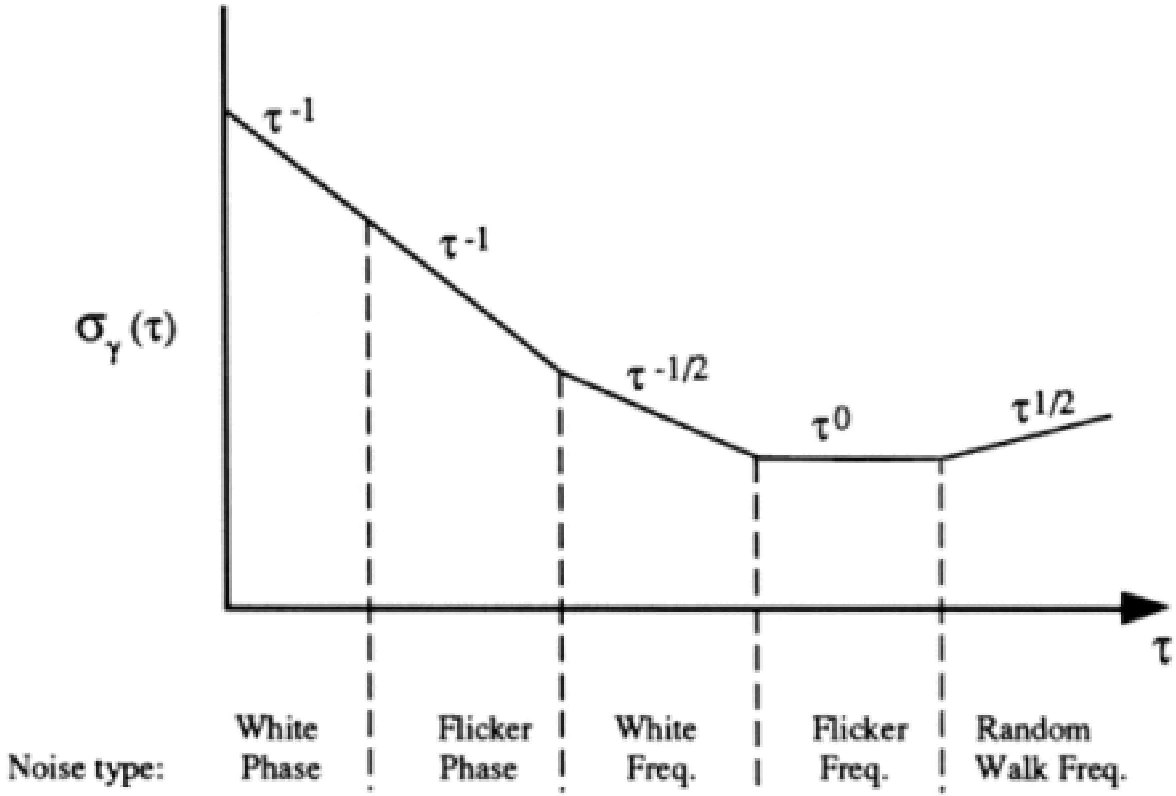
Exercise 3: Allan deviation for a linear frequency drift Consider an oscillator which frequency linearly changes in time $y(t) = at$, where a is the drift rate. Calculate the Allan deviation.

Since $\bar{y}_1 = [at_0 + a(t_0 + \tau)]/2$ and $\bar{y}_2 = [a(t_0 + \tau) + a(t_0 + 2\tau)]/2$ we will get

$$\sigma_y(\tau) = \left\langle a\tau/\sqrt{2} \right\rangle = \frac{a}{\sqrt{2}}\tau \quad . \tag{2.17}$$

Hence, linear frequency drift of an oscillator results in the Allan deviation linearly depending on averaging time τ.

Typical Allan deviation plot



Autocorrelation function

Stochastic process

$$B(t) = b(t) + \overline{B(t)}$$

Autocorrelation function

$$R_b(\tau) = \overline{b(t+\tau)b(t)} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} b(t+\tau)b(t)\,dt$$

Relation to dispersion

$$R_b(\tau = 0) = \sigma_b^2$$

Wiener-Khinchin theorem

$$
\begin{aligned}
R_b(\tau) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} a(\omega) e^{i\omega(t+\tau)}\, d\omega \int_{-\infty}^{\infty} a(\omega') e^{i\omega' t}\, d\omega'\, dt \\
&= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{it(\omega+\omega')}\, dt \right] a(\omega) a(\omega') e^{i\omega\tau}\, d\omega'\, d\omega
\end{aligned}
$$

$$
\begin{aligned}
R_b(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\omega) a(\omega') e^{i\omega\tau} \delta(\omega + \omega')\, d\omega'\, d\omega \\
&= \int_{-\infty}^{\infty} \frac{|a(\omega) a(\omega)|}{2\pi} e^{i\omega\tau}\, d\omega \\
&\equiv \int_{-\infty}^{\infty} S_b(f) e^{i2\pi f\tau}\, df\,.
\end{aligned}
$$

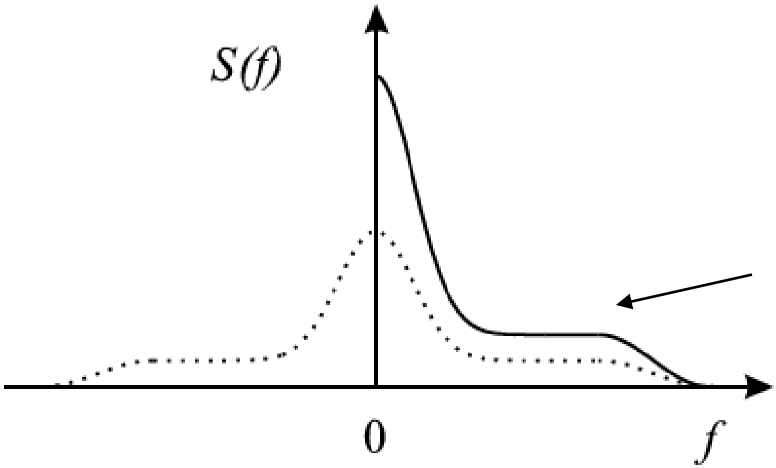
Dispersion:

$$
R_b(0) = \int_{-\infty}^{\infty} S_b(f)\, df
$$

Autocorrelation function if a Fourier transformation of a power spectral density!

ФИАН

Power spectrum of fluctuations



Units: V^2/Hz, Hz^2/Hz, etc.

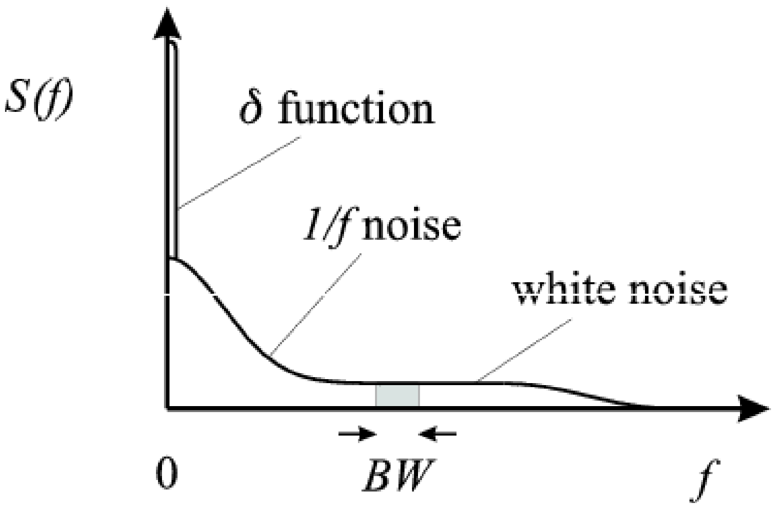
One- and two-sided spectral density

$$S_b^{\text{1-sided}}(f) = 2S_b^{\text{2-sided}}(f)$$

$$S_b^{\text{2-sided}}(f) \equiv \mathcal{F}^*\{R_b(\tau)\} = \int_{-\infty}^{\infty} R_b(\tau)\exp(-i2\pi f\tau)\,d\tau\,,$$

$$R_b(\tau) \equiv \mathcal{F}\{S_b^{\text{2-sided}}(f)\} = \int_{-\infty}^{\infty} S_b(f)\exp(i2\pi f\tau)\,df\,,$$

The spectral representation



$$R_\nu(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Delta\nu(t+\tau)\Delta\nu(t)\, dt$$

$$S_\nu^{\text{2-sided}}(f) = \int_{\infty}^{\infty} R_\nu(\tau) \exp(-i2\pi f \tau)\, d\tau$$

Frequency fluctuations spectral density

Relation between frequency and phase spectral densities

$$S_\nu(f) = f^2 S_\phi(f)$$

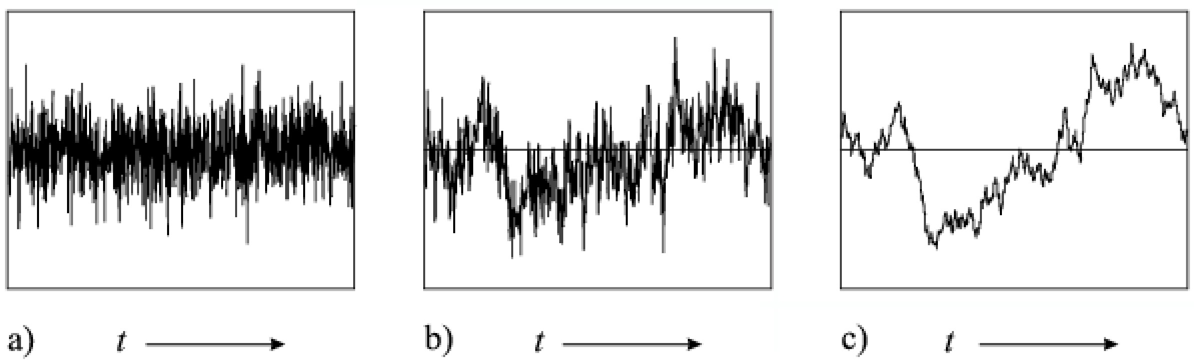
Frequency is the derivative of phase
then Fourier transformation

$$\int_0^{\infty} S_\nu^{\text{1-sided}}(f)\, df = \int_{-\infty}^{\infty} S_\nu^{\text{2-sided}}(f)\, df = \langle [\Delta\nu(t)]^2 \rangle = \sigma_\nu^2$$

ФИАН

Different types of fluctuation processes

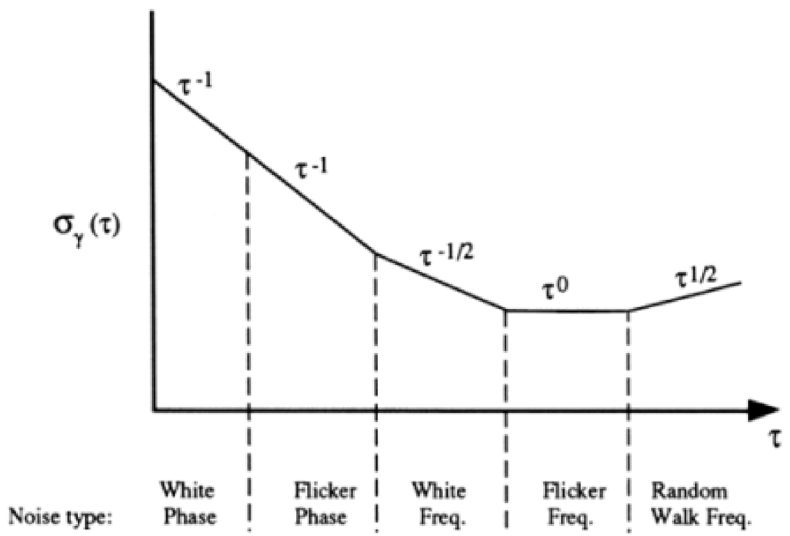
$$S_y(f) = \sum_{\alpha=-2}^{2} h_\alpha f^\alpha$$



White noise 1/f noise $1/f^2$ noise

Different types of fluctuation processes and corresponding Allan deviation

$S_y(f)$	$S_\phi(f)$	noise type	$\sigma_y^2(\tau)$
$h_{-2}f^{-2}$	$\nu_0^2 h_{-2} f^{-4}$	Frequency random walk	$(2\pi^2 h_{-2}/3)\tau^{+1}$
$h_{-1}f^{-1}$	$\nu_0^2 h_{-2} f^{-3}$	Frequency flicker noise	$2h_{-1}\ln 2\tau^0$
$h_0 f^0$	$\nu_0^2 h_0 f^{-2}$	Frequency white noise (phase random walk)	$(h_0/2)\tau^{-1}$
$h_1 f$	$\nu_0^2 h_1 f^{-1}$	Phase flicker noise	$h_1[1.038 + 3\ln(2\pi f_h \tau)] \cdot \tau^{-2}/4\pi^2$
$h_2 f^2$	$\nu_0^2 h_2 f^0$	Phase white noise	$[3h_2 f_h/(4\pi^2)]\tau^{-2}$



From spectral representation to time representation

The Allan deviation

$$\sigma_y^2(\tau) = \frac{1}{2}\langle(\bar{y}_2 - \bar{y}_1)^2\rangle = \frac{1}{2}\left\langle\left(\frac{1}{\tau}\int_{t_{k+1}}^{t_{k+2}} y(t')\,dt' - \frac{1}{\tau}\int_{t_k}^{t_{k+1}} y(t')\,dt'\right)^2\right\rangle$$

In the integral form

$$\sigma_y^2(\tau) = \left\langle \frac{1}{2}\left(\frac{1}{\tau}\int_t^{t+\tau} y(t')\,dt' - \frac{1}{\tau}\int_{t-\tau}^{t} y(t')\,dt'\right)^2\right\rangle$$

$$\sigma_y^2(\tau) = \left\langle\left(\int_{-\infty}^{\infty} y(t')h_\tau(t-t')\,dt'\right)^2\right\rangle$$

ФИАН

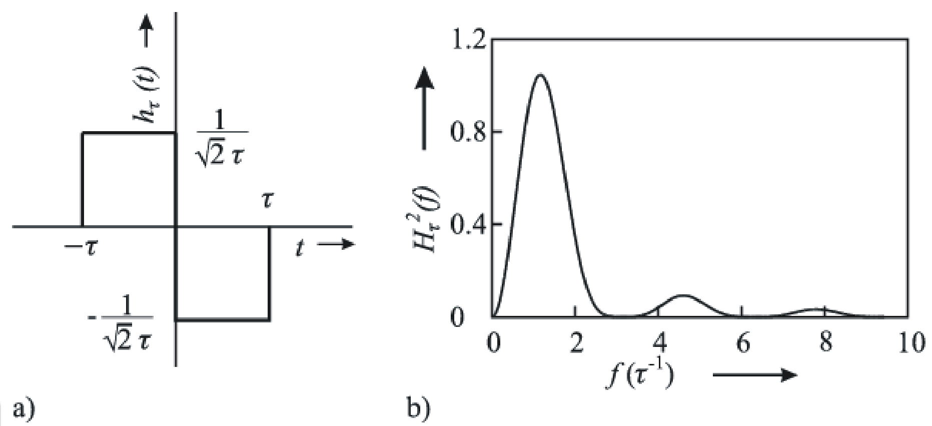
Transfer function :

$$\sigma_y^2(\tau) = \left\langle \left(\int_{-\infty}^{\infty} y(t')\, h_\tau(t - t')\, dt' \right)^2 \right\rangle$$

$$h_\tau(t) = \begin{cases} -\dfrac{1}{\sqrt{2}\tau} & \text{for} \quad -\tau < t < 0\,, \\ +\dfrac{1}{\sqrt{2}\tau} & \text{for} \quad 0 < t < \tau\,, \\ 0 & \text{for all oter cases} \end{cases}$$

$$\sigma_y^2(\tau) = \int_0^{\infty} |H_\tau(f)|^2 S_y^{\text{1-sided}}(f)\, df$$

$$H_\tau(f) = \mathcal{F}\{h_\tau(t)\}$$



Relation between power spectral density and Allan deviation

$$|H_\tau(f)|^2 = 2\frac{\sin^4(\pi f \tau)}{(\pi f \tau)^2}$$

$$\sigma_y^2(\tau) = 2\int_0^\infty S_y(f)\frac{\sin^4(\pi f \tau)}{(\pi f \tau)^2}\,df$$

Example: white phase noise

For example let us calculate Allan dispersion for phase white noise ($S_y = h_2 f^2$). Expression (2.46) gives:

$$\sigma_y^2(\tau) = 2\int_0^\infty h_2 f^2 \frac{\sin^4(\pi f \tau)}{(\pi f \tau)^2}\, df = \frac{2h_2}{\pi^2\tau^2}\int_0^\infty \sin^4(\pi f \tau)\, df\,. \qquad (2.47)$$

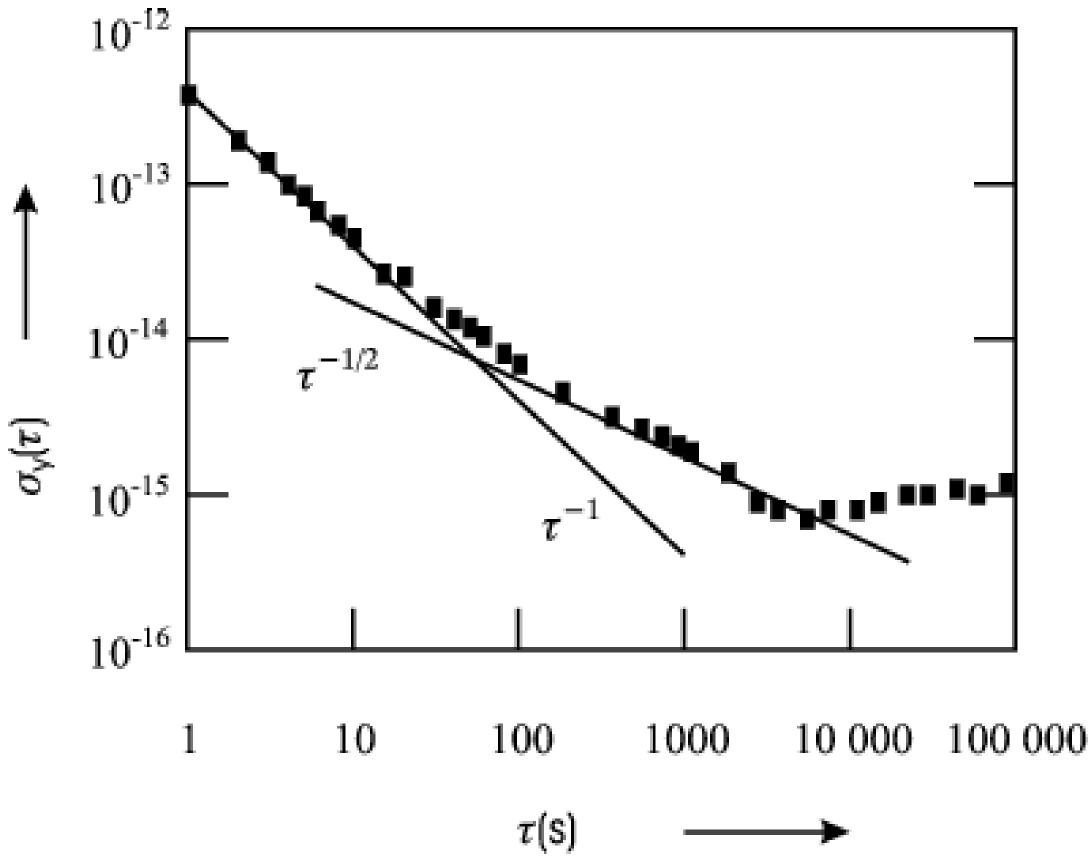
Integral in (2.47) does not converge at $f \to \infty$.

Cutoff frequency f_{h}!

$$\sigma_y^2(\tau) = \frac{2h_2}{\pi^2\tau^2}\int_0^{f_h} \sin^4(\pi f \tau)\, df = \frac{3h_2 f_h}{4\pi^2\tau^2} + \mathcal{O}(\tau^{-3})$$

Allan deviation in practice

Two hydrogen masers comparison



Allan deviations of oscillators in the H-lab